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Commutative algebra in group cohomology

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Abstract

We apply constructions from equivariant topology to Benson–Carlson resolutions and hence prove in (2.1) that the group cohomology ring of a finite group enjoys remarkable duality properties based on its global geometry. This recovers and generalizes the result of Benson–Carlson stating that a Cohen–Macaulay cohomology ring is automatically Gorenstein. We give an alternative approach to Tate cohomology of groups and in (4.1) show that the Tate cohomology of a group is close to being the cohomology of the projective space of the group cohomology ring.

0. Introduction

In this article we investigate the effect of certain standard constructions from commutative algebra when applied to the cohomology ring $H^*(G; k)$ of a finite group G with coefficients in a field k. This is a further step in understanding the connection between homological and commutative algebra, which is the algebraic counterpart of completion theorems and their duals from algebraic topology.

Our results come by using the methods from [6] in the context of group cohomology, and the present article will flesh out certain assertions made there. Fundamental to this application is the work of Benson–Carlson [1, 2] on algebraic analogues of free actions of finite groups on products of spheres.

The paper is divided into four sections. In Section 1 we recall some constructions from commutative algebra that we need. In Section 2 we prove that the group homology $H_*(G; M)$ is essentially the local cohomology of the corresponding cohomology $H^*(G; M)$ as a module over the graded local ring $H^*(G)$; this is an unusual duality phenomenon based on the global geometry of the ring $H^*(G)$. In Section 3 we explain how methods from topology [4, 8] give an alternative approach to the

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construction of Tate cohomology; this is simply homotopy theory of chain complexes over kG and may be of independent interest. Section 4 gives an analogue for Tate cohomology of the results of Section 2: the Tate cohomology $\hat{H}^*(G; M)$ is essentially the Čech cohomology of $H^*(G; M)$ as a module over $H^*(G)$. Geometrically speaking, this says that the Tate cohomology of the group is the cohomology (with all twists) of the projective space of the group cohomology ring $H^*(G)$.

1. Some commutative algebra

In the next section we shall make certain constructions that are analogous to classical constructions in commutative algebra. To establish notation and to ensure the reader is familiar with what we are modelling we give a summary of definitions. Further details and references can be found in [6].

Consider any commutative ring A and any finitely generated ideal $I = (\zeta_1, \ldots, \zeta_r)$. We may define the (flat) stable Koszul complex

$$K^\bullet(I^\infty) = \{(A \to A[1/\zeta_1]) \otimes (A \to A[1/\zeta_2]) \otimes \cdots \otimes (A \to A[1/\zeta_r])\}.$$

It is not hard to check that up to quasi-isomorphism this is independent of the generators and depends only on the radical of I. We then define the local cohomology groups of the A-module M by

$$H_I^*(A; M) := H^*(K^\bullet(I^\infty) \otimes M).$$

When the ring A is clear from the context it is omitted from the notation. A theorem of Grothendieck states that, if A is Noetherian, local cohomology calculates the right derived functors of the left exact I-power torsion functor $M \mapsto \Gamma_I(M) = \{m \in M \mid I^k m = 0 \text{ for } k \text{ sufficiently large}\}$. For an elementary approach directly applicable to graded rings see [5]. Hence, for instance

$$H_I^n(M) = \varinjlim_k \operatorname{Ext}^n(A/I^k, M).$$

Of course if A is a graded ring and I is a homogeneous ideal the local cohomology sgroups of any graded module are graded.

We shall also need the flat Čech complex $\check{C}_f^\bullet(I)$ which may be constructed by taking the Koszul complex, deleting the A in codegree zero, and regrading. Equivalently there is a fibre sequence

$$K^\bullet(I^\infty) \to A \to \check{C}_f^\bullet(I). \tag{1}$$

The Čech cohomology groups of a module M are then defined by

$$\check{H}^*(A; M) = H^*(\check{C}_f^\bullet(I) \otimes M).$$

This can be regarded as the use of the cover of $Spec(A) \setminus V(I)$ by open affine sets $Spec A[1/\zeta_i]$ to calculate its cohomology with coefficients in the sheaf defined by M. As with local cohomology, Čech cohomology calculates the right derived functors of the global sections functor, and it vanishes in codegrees at or above the dimension of A.

2. Group homology and local cohomology of the cohomology ring

Let G be a finite group and let k be a field. We shall be applying the constructions of Section 1 when A is the group cohomology ring $H^*(G) = H^*(G; k)$ and I is its maximal ideal of all elements of positive codegree.

Theorem 2.1. *If M is a kG-module or a bounded below complex of kG-modules, there is a spectral sequence*

$$E_2^{s,t} = \{H_I^s(H^*(G; M))\}^t \Rightarrow H_{-s-t}(G; M)$$

with differentials $d_u : E_u^{s,t} \to E_u^{s+u,t-u+1}$.

Before beginning work on a proof we consider its implications.

First take $M = k$ and note that, since the homology $H_*(G)$ is k-dual to the cohomology ring, Theorem 2.1 is in effect the statement that local cohomology dualises the ring, and in the Cohen–Macaulay case this is precisely true.

Corollary 2.2. *If $H^*(G)$ is Cohen–Macaulay then*

$$H_n(G) = \{H_I^r(H^*(G))\}^{-n-r}.$$

In topological notation we may say

$$H_*(G) = \Sigma^r H_I^r(H^*(G)),$$

where the suspension is cohomological.

We thus recover an observation of Benson–Carlson.

Corollary 2.3 (Benson–Carlson [2]). *If $H^*(G)$ is Cohen–Macaulay then it is also Gorenstein.*

Proof. Indeed we see from Corollary 2.2 that the top local cohomology is the injective module $H_*(G) = \{H^*(G)\}^*$ (indeed it is the injective envelope of k). This characterises Gorenstein rings amongst Cohen–Macaulay rings since the injective envelope of the residue field is a dualising module [9, (4.10) and (4.14)] (the graded case is made explicit in [3]). □

It seems also worthwhile making explicit the almost Cohen–Macaulay case when the only nontrivial local cohomology groups are in codegrees $r-1$ and r.

Corollary 2.4. *If $H^*(G;M)$ is an almost Cohen–Macaulay module then there is a short exact sequence*

$$0 \to \Sigma^r H_I^r(H^*(G;M)) \to H_*(G;M) \to \Sigma^{r-1} H_I^{r-1}(H^*(G;M)) \to 0.$$

We now set about constructing the complexes used in the proof of Theorem 2.1.

For a homogeneous element $\zeta \in H^n(G) = \mathrm{Ext}^n(k,k)$ we may form a corresponding complex of kG modules

$$C_\zeta = (C_{n-1}(\zeta) \to C_{n-2} \to \cdots \to C_1 \to C_0), \tag{2}$$

where C_i is projective for $0 \le i \le n-2$ and where C_ζ has the homology of an $(n-1)$-sphere S^{n-1}. The module $C_{n-1}(\zeta)$ (which is not usually projective) may be constructed using the pushout square

$$\begin{tikzcd} \Omega^n k \arrow[r] \arrow[d, "\zeta"'] & C_{n-1} \arrow[d] \\ k \arrow[r] & C_{n-1}(\zeta) \end{tikzcd}$$

where $\Omega^n k$ is the kernel of $C_{n-1} \to C_{n-2}$. Now by Noether's normalization theorem, if r is the Krull dimension of $H^*(G)$ then there are algebraically independent elements $\zeta_i \in H^{n_i}(G)$ for $i = 1, 2, \ldots, r$ so that $H^*(G)$ is finitely generated over $k[\zeta_1, \ldots, \zeta_r]$. Note in particular that the radical of the ideal $(\zeta_1, \ldots, \zeta_r)$ is therefore the ideal I of elements of positive codegree. Now take tensor products to obtain the chain complex $B = C_{\zeta_1} \otimes \cdots \otimes C_{\zeta_r}$ with the homology of $S^{n_1-1} \times \cdots \times S^{n_r-1}$. Benson and Carlson use the theory of support varieties to prove that B is a complex of projectives. It is convenient to think of B as graded over $\mathbb{Z}^r$, and concentrated in cuboidal box with vertices at $(\varepsilon_1(n_1-1), \ldots, \varepsilon_r(n_r-1))$ where $\varepsilon_i = 0$ or 1. Now we may form various complexes by stacking these cuboids, suitably shifted, to fill up regions of $\mathbb{Z}^r$; the differential joining adjacent boxes is obtained by splicing two copies of C_ζ to form

$$(C_{n-1}(\zeta) \to C_{n-2} \to \cdots \to C_1 \to C_0) \to (C_{n-1}(f) \to C_{n-2} \to \cdots \to C_1 \to C_0)$$

using the fact that $H_0(C_\zeta) = k = H_{n-1}(C_\zeta)$. In particular, we let R denote the multicomplex obtained by filling up the nonnegative orthant. Its total complex is a projective resolution of k. We also let $R^!$ denote the multicomplex obtained by filling up the negative orthant, and note that since finitely generated projectives are injective, its total complex is a shifted copy of an injective resolution of k. Now if $\sigma \subseteq \{1, 2, \ldots, r\}$ let $R[\sigma]$ denote the result of filling up the region specified by requiring $n_i \ge 0$ if $i \notin \sigma$. Notice that this is the inverse limit of shifts of R under multiplication by $\zeta'_\sigma := \prod_{i \in \sigma} \zeta'_i$. Hence, because the limit is achieved in each degree, if M is any kG-module

$$\mathrm{Hom}(R[\sigma], M) = \varinjlim(\mathrm{Hom}(R, M), \zeta_\sigma) = \mathrm{Hom}(R, M)[1/\zeta_\sigma]. \tag{3}$$

More generally (3) holds if M is any chain complex of kG-modules which is bounded below in the sense that the terms are zero in sufficiently negative degrees. Note that the cohomology of the G-invariants of $\mathrm{Hom}(R, M)$ is the group cohomology $H^*(G; M)$. Now observe that when $\sigma \subseteq \tau$ there is a projection $R[\tau] \to R[\sigma]$. Thus we may form the dual stable Koszul chain complex

$$L_\bullet = \left(\bigoplus_{|\sigma| = r} R[\sigma] \to \bigoplus_{|\sigma| = r-1} R[\sigma] \to \cdots \to \bigoplus_{|\sigma| = 1} R[\sigma] \to \bigoplus_{|\sigma| = 0} R[\sigma] \right), \tag{4}$$

where the differential from $R[\sigma]$ is the alternating sum of the projections corresponding to the maximal faces of σ. Now that we have two types of grading we shall refer to the grading arising from projective resolutions as the P-grading, and that arising from Koszul complexes as the K-grading. The following lemma describes the cohomology in the Koszul direction.

Lemma 2.5. (i) *All cycle, boundary and homology groups are kG-projective.*
(ii) $H_i(L_\bullet) = 0$ *for* $0 \le i \le r-1$.
(iii) $H_r(L_\bullet)$ *is the multicomplex* $R^!$ *obtained by stacking boxes in the negative orthant.*

Proof. For each point of $\mathbb{Z}^r$ there is a corresponding projective P which will be placed there if all of $\mathbb{Z}^r$ is filled with boxes. Now at this point each L_i is a sum of copies of P. Consider the orthant O_ρ which is negative on the ith coordinate if and only if $i \in \rho$. In this orthant L_i is a sum with one copy of P for each σ with i elements which contains ρ. By the binomial theorem this is exact unless $\rho = \{1, 2, \ldots, r\}$. □

Corollary 2.6. *The total complex* $H_r(L_\bullet)$ *provides a resolution*

$$0 \to k \to I_0 \to I_1 \to \cdots$$

of k *by finitely generated injectives, graded so that the zeroth term is in P-degree* $-r$. □

The proof of Theorem 2.1 is now straightforward.

Proof of Theorem 2.1. For each K-degree i, form the total complex of $\mathrm{Hom}(L_i, M)$ and let $\mathrm{Hom}(L_i, M)^j$ the term in P-degree j. Now consider the two spectral sequences converging to the cohomology of the total complex of the resulting double cochain complex $\mathrm{Hom}(L_\bullet, M)^G$ of G-invariants.

Consider what happens if we take cohomology first in the Koszul (i) direction. Since all cycle, boundary and homology groups are kG projective, passage to G fixed points commutes with passage to K-cohomology, which is thus $\mathrm{Hom}_{kG}(H_*(L_\bullet), M)$. By Lemma 2.5 this is concentrated in K-degree r where it is $\mathrm{Hom}_{kG}(I^\bullet, M)$. Thus the resulting spectral sequence collapses at the E_2 page. Since $\mathrm{Hom}_{kG}(kG, M) \cong (kG)^* \otimes_{kG} M$, naturally in kG, we see that this E_2 term is $H_*(G; M)$.

On the other hand, we may first take cohomology in the projective (j) direction. By (3), since $H^*(\{\mathrm{Hom}(R, M)\}^G) = H^*(G; M)$ we obtain the stable Koszul complex for $H^*(G; M)$ associated to the sequence of elements $\zeta_1, \ldots, \zeta_r$. The K-cohomology of this is the local cohomology of $H^*(G; M)$ at the ideal $(\zeta_1, \dots, \zeta_r)$. We have already noted that this ideal has radical I. $\square$

3. Warwick duality

In this section the field k may be replaced by $\mathbb{Z}$ at the expense of inserting the hypothesis that certain complexes are $\mathbb{Z}$-free; we leave this to the interested reader. In the subsequent discussion, all hom's and tensor products are over k unless otherwise specified. When we say a module is projective, we refer to its structure as a kG-module. All chain complexes are graded over the integers, and we say a chain complex $C_\bullet$ is bounded below if there is an n_- so that $C_n = 0$ for $n < n_-$. We say that a cochain complex $D^\bullet$ is bounded below if the associated chain complex $D_\bullet$ defined by $D_n = D^{-n}$ is. Similarly we say $C_\bullet$ is bounded above if there is an n_+ so that $C_n = 0$ for $n > n_+$. A chain complex $D_\bullet$ is weakly contractible if the associated fixed point subcomplexes $D_\bullet^H$ are exact for all subgroups H. A weakly contractible, bounded below complex of projectives is contractible. A map of chain complexes is a weak equivalence if its mapping cone is weakly contractible. Weak equivalence is the equivalence relation generated by weak equivalences: thus if $C_\bullet$ and $D_\bullet$ are weakly equivalent then for all subgroups H the complexes $C_\bullet^H$ and $D_\bullet^H$ are quasi-isomorphic, and in particular they have the same homology. We freely use terminology from the homotopy theory of chain complexes; relevant summaries are given in [7]. We shall continue to write using homology in the conventional way, but the parallels with equivariant topology are made clearer by noting that $H^0(\mathrm{Hom}_{kG}(C_\bullet, D_\bullet)) = [C_\bullet, D_\bullet]^G$ where the right-hand side denotes equivariant chain homotopy classes of equivariant chain maps.

Consider any resolution $P_\bullet$ of the ring k by finitely generated projective kG-modules and let $\varepsilon: P_\bullet \to k$ be the augmentation. By definition of $P_\bullet$ this is a homology isomorphism. Let us define $\tilde{P}_\bullet$ to be the mapping cone of ε. Note that $\tilde{P}_\bullet$ is exact but is not a complex of projective modules. The cofibre sequence

$$P_\bullet \to k \to \tilde{P}_\bullet \tag{5}$$

is precisely analogous to (1), and equally fundamental.

Given a complex M of kG-modules, we may form the associated co-projective complex $c(M) = \mathrm{Hom}_k(P_\bullet, M)$ and the projective complex $f(M) = P_\bullet \otimes M$ (the notation comes from [8], where c stands for 'cofree' or 'completion', and f stands for 'free'). Note that if M is bounded below so is $f(M)$, but $c(M)$ will never be bounded below unless $M = 0$.

Lemma 3.1. *If E is exact (for example if $E = \tilde{P}_\bullet$) and F is a bounded below complex of projective modules then $F \otimes E$ and $\mathrm{Hom}(F, E)$ are weakly contractible. Hence in particular $c(E)$ and $f(E)$ are weakly contractible.*

Proof. The case $F = kG$ is a routine exercise: we simply choose a k-contracting homotopy of E and extend it to a kG-contracting homotopy of $kG \otimes E \cong \mathrm{Hom}(kG, E)$. *The case when* F is concentrated in a single degree follows easily by properties of products. The case when F is concentrated in a finite range of degrees follows by induction and the five lemma, and the general case follows since any bounded below F is a direct limit of such complexes. □

Corollary 3.2. *The augmentation* ε *induces a weak equivalence*

$$f(c(M)) = \mathrm{Hom}(P_\bullet, M) \otimes P_\bullet \xrightarrow{\simeq} \mathrm{Hom}(k, M) \otimes P_\bullet = f(M).$$

Proof. Since $\mathrm{Hom}(\ , M) \otimes P_\bullet$ preserves cofibre sequences, it is enough to apply Lemma 3.1 to the exact complex $\mathrm{Hom}(\tilde{P}_\bullet, M)$. □

We therefore have two obvious choices for a norm map $f(M) \to c(M)$, and we observe that they are equivalent.

Lemma 3.3. *The square*

$$\begin{tikzcd} \mathrm{Hom}(k, M) \otimes P_\bullet \arrow[r, "\varepsilon^* \otimes \varepsilon"] \arrow[d, "\varepsilon^* \otimes 1"'] & \mathrm{Hom}(P_\bullet, M) \otimes k \arrow[d, "1 \otimes 1"] \\ \mathrm{Hom}(P_\bullet, M) \otimes P_\bullet \arrow[r, "1 \otimes \varepsilon"] & \mathrm{Hom}(P_\bullet, M) \otimes k \end{tikzcd}$$

is commutative. □

Combining Corollary 3.2 and Lemma 3.3 we have the desired conclusion.

Corollary 3.4. *The mapping cones of the top and bottom horizontals in* Corollary 3.2 *are weakly equivalent.* □

Since mapping cones commute with tensor products it is easy to identify the mapping cone of the bottom row as

$$t(M) := \mathrm{Hom}(P_\bullet, M) \otimes \tilde{P}_\bullet. \tag{6}$$

Following the pattern of [4, 8] we refer to $t(M)$ as the Tate complex associated to M.

Consider then the meaning of Corollary 3.4 in the case when $M = k$. Classically the Tate resolution $T_\bullet$ of G is obtained by splicing together projective and injective resolutions of k by the norm map; since the dual $\mathrm{Hom}(P_\bullet, k)$ of $P_\bullet$ is an injective resolution of k, one way doing this is to take $T_\bullet$ to be the mapping cone of $N = \varepsilon^* \otimes \varepsilon$. Thus by Corollary 3.4 we have a weak equivalence

$$T_\bullet \simeq P_\bullet^* \otimes \tilde{P}_\bullet. \tag{7}$$

We next recall the classical definition of the Tate homology and cohomology with coefficients in a chain complex M. Here we suppose given any chain complex M of kG-modules and we take

$$\hat{H}_*(G;M) = H_*(T_\bullet \otimes_{kG} M) \quad \text{and} \quad \hat{H}^*(G;M) = H^*(\mathrm{Hom}_{kG}(T_\bullet^*, M)).$$

In the case of homology, some authors use $H_*(\Sigma^{-1} T_\bullet \otimes_{kG} M)$ on the grounds that there is a map $\Sigma^{-1} T_\bullet \to P_\bullet$ of degree 0 which can be used to compare Tate homology and ordinary homology. The present convention is chosen because the ring structure in cohomology forces the choice of grading and the homology grading gives $\hat{H}_n(G;M) = \hat{H}^{-n}(G;M)$ without a shift in degree for any complex M which is bounded above and below: this argument is irresistable in the topological context [8].

On the other hand, we may form the homology and cohomology groups

$$H_*(G;M) = H_*(P_\bullet \otimes_{kG} M) \quad \text{and } H^*(G;M) = H^*(\mathrm{Hom}_{kG}(P_\bullet, M)).$$

It is useful to bear in mind the fundamental distinction between cohomology with coefficients in a chain complex (as defined above), and the equivariant cohomology of chain complexes, defined by $H_G^*(M) = H^*(\mathrm{Hom}_{kG}(P_\bullet \otimes M, k))$ or $\hat{H}_G^*(M) = H^*(\mathrm{Hom}_{kG}(T_\bullet \otimes M, k))$. We make no use of the latter notion here, but the distinction clarifies the relationship between the present discussion and the topological case.

With this notation we may relate $t(M)$ to the classical Tate homology and cohomology with coefficients in M. In the general case we regard the definition in terms of $t(M)$ as the more useful and well behaved.

Proposition 3.5. (a) *If M is bounded above then*

$$H_*(t(M)//kG) = \hat{H}_*(G;M).$$

(b) *If M is bounded below then*

$$H^*(t(M)^G) = \hat{H}^*(G;M).$$

Proof. (a) We note below that provided M is bounded above $t(M) = \mathrm{Hom}(P_\bullet, M) \otimes \tilde{P}_\bullet$ is isomorphic to $\mathrm{Hom}(P_\bullet, k) \otimes M \otimes \tilde{P}_* = P_\bullet^* \otimes \tilde{P}_\bullet \otimes M$; by (7) this is weakly equivalent to $T_\bullet \otimes M$.

Indeed there is a natural map

$$\mathrm{Hom}(P_\bullet, k) \otimes M \to \mathrm{Hom}(P_\bullet, M).$$

In degree n this is the natural map

$$\bigoplus_{j-i=n} (P_i)^* \otimes M_j \to \prod_{j-i=n} \mathrm{Hom}(P_i, M_j),$$

which is isomorphic provided the sum and product are finite, which happens precisely if M is bounded above. $\square$

The proof given above only proves part (b) when M is bounded above and below. The correct level of generality involves a change of viewpoint; we give the proof of part (b) at the end of the section.

Because the final functor in the definition of $t(M)$ is a tensor product this complex is best suited for use in homology and more generally for use in $t(M) \otimes_{kG} M'$. The good formal properties of Tate theory derive from the existence of a second form which is better suited for cohomology and more generally for use in $\mathrm{Hom}_{kG}(M', t(M))$. This cohomological avatar has a Hom as the final functor.

Proposition 3.6 (Warwick duality). *There is a weak equivalence*

$$\mathrm{Hom}(\tilde{P}_\bullet, M \otimes \Sigma P_\bullet) \simeq \mathrm{Hom}(P_\bullet, M) \otimes \tilde{P}_\bullet.$$

Proof. We have the following string of weak equivalences induced by maps from the cofibre sequence (5).

$$\begin{aligned}
\mathrm{Hom}(\tilde{P}_\bullet, M \otimes \Sigma P_\bullet) &\simeq \mathrm{Hom}(\tilde{P}_\bullet, \mathrm{Hom}(P_\bullet, M) \otimes \Sigma P_\bullet) && \text{(by Corollary 3.2)} \\
&\simeq \mathrm{Hom}(\tilde{P}_\bullet, \mathrm{Hom}(P_\bullet, M) \otimes \tilde{P}_\bullet) && \text{(by Lemma 3.1)} \\
&\simeq \mathrm{Hom}(k, \mathrm{Hom}(P_\bullet, M) \otimes \tilde{P}_\bullet) && \text{(by Lemma 3.1)} \\
&= \mathrm{Hom}(P_\bullet, M) \otimes \tilde{P}_\bullet \quad \square
\end{aligned}$$

In case $M = k$ Warwick duality is a very familiar fact.

Corollary 3.7.

$$(T_\bullet)^* \simeq \Sigma^{-1} T_\bullet$$

Proof by Warwick duality. Since $P_\bullet$ is finitely generated in each degree we have $\mathrm{Hom}(\tilde{P}_\bullet \otimes (\Sigma P_\bullet)^*, k) = \mathrm{Hom}(\tilde{P}_\bullet, \Sigma P_\bullet)$. Writing down Warwick duality gives

$$\Sigma^{-1}(\tilde{P}_\bullet \otimes P_\bullet^*)^* \cong \mathrm{Hom}(\tilde{P}_\bullet, \Sigma P_\bullet) \simeq \mathrm{Hom}(P_\bullet, k) \otimes \tilde{P}_\bullet = (P_\bullet)^* \otimes \tilde{P}_\bullet,$$

from which the result follows by (7). $\square$

Conventional proof. One method is to use some characterization of $T_\bullet$, but the following is more precise. The Tate complex $T_\bullet$ is the cofibre of the norm map $N: P_\bullet^* \to P_\bullet$. Since $N^* = N$, and the fibre is the desuspension of the cofibre, the result follows. $\square$

Proof of Proposition 3.5(b) We note that, as in the proof of part (a), if M is bounded below then $\mathrm{Hom}(\tilde{P}_\bullet, M \otimes \Sigma P_\bullet)$ is isomorphic to $\mathrm{Hom}(\tilde{P}_\bullet \otimes (\Sigma P_\bullet)^*, M)$; by (7) and Corollary 3.7 this is weakly equivalent to $\mathrm{Hom}(T_\bullet, M)$. $\square$

4. Tate cohomology and local Tate cohomology of the cohomology ring

We now want to repeat the programme of Section 2 for Tate cohomology.

Theorem 4.1. *If M is a chain complex which is bounded above and below there is a spectral sequence*

$$E_2^{s,t} = \check{H}_I^s(H^*(G;M))^t \Rightarrow \hat{H}^{s+t}(G;M).$$

Remarks. (a) Since this Čech cohomology group is the cohomology of $Proj\,(H^*(G))$ with coefficients in the sheaf arising from $H^*(G;M)$ we reach the statement that the Tate cohomology of G is essentially the cohomology of the projective space of $H^*(G)$.

(b) It seems likely that the boundedness hypotheses in the theorem can be weakened by use of $t(M)$ and projective approximations parallelling those of [6]. This involves obscuring the argument with a good deal of extra notation and circumlocution to avoid hom's and tensor products over R^*.

First, notice that in the dual stable Koszul complex (4), $L_0 = R$. Thus we may form a dual analogue of (1), defining the dual Čech complex $D_\bullet$ by the cofibre sequence

$$D_\bullet \to R \to L_\bullet. \tag{8}$$

We therefore have the following immediate consequence of Lemma 2.5.

Lemma 4.2. (i) *All cycle, boundary and homology groups in* (8) *are kG-projective.*
(ii) $H_i(D_\bullet) = 0$ *unless* $i = 0$ *or* $i = r-1$.
(iii) $H_0(D_\bullet) = R$.
(iv) $H_{r-1}(D_\bullet) = R^!$.

We remarked in (3) that $R[\sigma]^* = R^*[1/\zeta_\sigma]$, so it is clear that the dual of $D_\bullet$ is like the flat Čech complex.

We now view Lemma 2.5 in a different way. In fact we see that the K-complex with $R^!$ in K-degree r and 0 elsewhere is a subcomplex of $L_\bullet$. Thus we have the inclusion

$$\Sigma^r R^! \to L_\bullet$$

of total complexes, which Lemma 2.5(iii) states is a homology isomorphism; by Lemma 2.5(i) it is a weak equivalence. We want to say something similar for Lemma 4.2.

Lemma 4.3. *There is a chain homotopy commutative diagram*

$$\begin{tikzcd} L_\bullet \arrow[d] & R \arrow[l] \\ \Sigma^r R^! & R \arrow[l] \arrow[u, "="'] \end{tikzcd}$$

of total complexes in which the lower horizontal is the dual of the norm map. Furthermore, the verticals are weak equivalences.

Proof. We only need to verify that two chain maps from R to $L_\bullet$ are chain homotopic, but since R is a bounded below projective approximation to k such maps are classified by their effect in H_0. Since the left-hand vertical is a homology isomorphism by Lemma 2.5, it is enough to observe that the top horizontal induces a nontrivial map of k in H_0. Since the left-hand vertical is a homology isomorphism by Lemma 2.5, it is enough to observe that the top horizontal induces a nontrivial map of k in H_0. We recommend the reader now draws the picture for $r = 1$ if he has not done so already.

In other words we need to show that if $x \in L_0$ represents a nontrivial class in $H_0(L_0) = k$ then it is still not a cycle in the total complex. Consider $y_i \in R[\{i\}]_0 \subseteq (L_1)_0$; its Koszul boundary is equal to that of its component in multidegree $(0, \ldots, 0)$, so it is sufficient to consider the case when y_i is in this multidegree. But if the Koszul boundary of y_i is nontrivial in homology it also maps nontrivially in the P-direction, since this was the splicing used in the construction of $R[\{i\}]$. $\square$

We may now dualize the diagram Lemma 4.3 and so obtain a homotopy commutative square, from which we may construct a map of cofibre sequences.

Corollary 4.4. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
(L_\bullet)^* \arrow[r] \arrow[d] & R^* \arrow[r] \arrow[d, "="'] & (D_\bullet)^* \arrow[d] \\
(\Sigma^r R^!)^* \arrow[r, "N"] & R^* \arrow[r] & T_\bullet
\end{tikzcd}$$

in which the rows are cofibre sequences and the columns are weak equivalences.

Proof. The construction of the diagrams from Lemma 4.3 was explained above. The left-hand verticals in Corollary 4.4 are therefore weak equivalences since homology commutes with k-duality, and the right-hand vertical is a weak equivalence by the 5-lemma. $\square$

This gives us all the ingredients to prove our result about Tate homology.

Proof of Theorem 4.1. We consider the cochain complex

$$U^\bullet(M) = M \otimes (D_\bullet)^*.$$

By Corollary 4.4 this is equivalent to $M \otimes T_\bullet$. By the boundedness of M this is isomorphic to $\mathrm{Hom}(T_\bullet^*, M)$, and the cohomology of its invariants is Tate cohomology with coefficients in M.

On the other hand, we may consider $U^\bullet(M)$ as a double complex using Koszul and projective gradings. Now $(D_\bullet)^*$ is a sum of terms $R^*[1/\zeta_\sigma]$. and so, since tensor products commute with direct limits, $U^\bullet(M)$ is a sum of terms $(M \otimes R^*)[1/\zeta_\sigma]$. Since M is bounded above $M \otimes R^* \cong \mathrm{Hom}(R, M)$. Hence the P-cohomology of the G-invariants is $H^*(G; M) \otimes_A \check{C}_f^\bullet(\mathrm{I})$; the K-cohomology of this is the indicated E_2 term. □

Remark 4.5. One consequence is that we have a natural weak equivalence of represented functors

$$\mathrm{Hom}(\bullet, U^\bullet(M)) \to \mathrm{Hom}(\bullet, t(M)).$$

This is directly analogous to the theorem about equivariant cohomology theories [6, A.4] stating that the local cohomology theorem (analogous to Theorem 2.1) implies that Tate cohomology is equal to local Tate cohomology of [6]. Of course local Tate cohomology made no explicit appearance in the above discussion. The reason is that if M is bounded above, $H^*(G; M)$ is bounded above and hence it is visibly I-adically complete. By [7, 4.1] the spectral sequence of [6, 3.3] collapses to show that the local Tate cohomology is the Čech cohomology of $H^*(G; M)$.

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